# Optimal Retirement Age and Ageing Population*

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**Abstract**

We study the effect of ageing population on the optimal retirement age that maximizes social welfare. An increase in longevity reduces per capita labor and raises the optimal retirement age. A drop in fertility increases the weight of seniors, reducing per capita labor, but diminishes the weight of children and raises the weight of more experienced workers, generating the opposite effect on per capita labor. Despite of these offsetting mechanisms, we provide a precise yardstick to determine the net effect of a drop in fertility on both labor supply and optimal retirement, which can be easily obtained from the data.

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## 1. Introduction

The fall in fertility and the continuing rise in longevity have led to a significant increase in the proportion of the older population in most developed countries. The increasing concern about the possible negative consequences of ageing population over social welfare has generated a public debate about the political measures to be taken to alleviate these effects. Such measures are central objectives of current political economic agendas of most governments. Policy reforms, such as the increase in the legal retirement age or the adoption of measures devoted to increase fertility rates, have been considered, but the question is whether the increase in the retirement age is an efficient response to the ageing population process. Are really pronatalist policies the right solution? The answers to these questions are more complex than a priori appear. As we will show, the results that we obtained seem to be running against widespread public opinion and some of the current policy reforms; in fact, these reforms do not take into account some important mechanisms brought by our analysis.

On July 7th, 2010, newspapers around the world echoed the latest report from the European Commission that suggested there was a need to raise the average retirement age in the 27-nation EU bloc from the current age of 65 up to 70 by 2060 if workers were to continue supporting retirees at current rates. As of today, there are four working-age people for every person over 65 in the 27-nation. Official estimations predict that this ratio will drop to two for every person over 65 by 2060. The concern about the effects of ageing population is not a new issue [1] neither something exclusive to Europe.[2] Throughout recent years developed countries have implemented a range of measures devoted to raising the fertility rate in order to slow down the aging of the population and to increase the ratio of workers to retirees. Even countries that could not be labeled as pronatalist, like the United States, have developed many family or social policies hopping that they generate an increase

[1] In recent years Spain, Germany, Netherlands and Denmark have deferred the retirement age to 67, while United Kingdom has delayed the retirement age to 68.

[2] See for example, Velloso (2005), Weller (2005), Boldrin et al. (2000), OECD (1998), European Commission Report (2004) and Congress of the United Status Report (2004).

in fertility.[3] However, we show in the current paper that a higher fertility rate is not necessarily the solution. We provide a clear yardstick to determine whether pronatalist policies have a positive or negative effect on the optimal retirement age.

This paper develops a neoclassical growth model to study the optimal retirement age that maximizes social welfare. Two relevant reasons motivate this analysis: first, what really matters in the ageing population debate are the economic implications of the ageing population process on the social welfare in the economy; and second, the set of results derived from the welfare analysis we present here is the most reliable source for political advice since it would shed light on the targets that economic policy should seek. We show that the optimal allocation may be decentralized through an optimal social security system. The target of economic policy should be to implement an optimal social security system which delivers an efficient allocation. Thus, the present analysis is the most suitable for clarifying the goal that economic policy should pursue and the type of social security system that should be implemented.

The model is designed to analyze how changes in demographic variables, such as fertility rate and life expectancy, affect the optimal retirement age. The model reveals that in determining the optimal retirement age, the role played by a drop in the fertility rate is different from the role played by an increase in longevity. While an increase in longevity always implies an increase in the optimal retirement age, a drop in the fertility rate does not. This paper provides an accurate measure to determine the sign of the effect of a drop in the fertility rate over per capita labor and the optimal retirement age. This is a relevant contribution since this measure, which has the additional advantage that may be easily obtained from the data, provides a precise criterion for political advice in pronatalist policies and helps to clarify the debate over the retirement age and to answer the questions raised above.

This paper introduces a framework in which agents live during three stages of different length: childhood, youth, and old age. Births grow at a constant rate. While during childhood and youth agents survive for sure, during old age the probability of surviving decreases with age, being zero at a certain threshold. Agents

[3]In this respect Grant et al. (2006) identify several forms of policy interventions in family life or population structure. See Kohler et al. (2006) for the case of Europe, Suzuki (2004) for Japan and Fustos (2010) for the United States.

work during the youth and they may retire before arriving to old age. Thus, retirement age is an endogenous variable. While the amount of efficiency units of labor of young agents varies with age due to experience, the subjective cost of working increases with age. The optimal retirement age maximizes a social welfare function. The paper analyzes the effect of an ageing population on it, distinguishing between an increase in longevity and a drop in the fertility rate.

A rise in longevity (an increase in the probability of surviving) enlarges the weight of old agents over total population and the dependency ratio, defined as the non working population (children and retirees) over working population. Thus, a rise in longevity reduces the activity rate, defined as working population over total population. Since old agents do not work, an increase in longevity implies a drop in per capita labor, which works as a negative wealth effect. The consequence of the negative wealth effect is a higher optimal retirement age.

The effect of a decrease in fertility over both per capita labor and optimal retirement age is much more complex since it involves some offsetting mechanisms. When the fertility rate decreases, the weight that younger people have over total population goes down, decreasing the weight of children which reduces the dependency ratio, while the weight that older people have in population goes up, increasing the dependency ratio. Thus, the effect of a drop in fertility over the dependency ratio and the activity rate is ambiguous. It turns out that the effect of a drop in fertility over the activity rate is positive if the average age of workers is greater than the average age of total population. This measure can be easily obtained from the data. However, the fact that a drop in fertility may produce an increase in the activity rate does not imply a clear conclusion about the effect of a drop in fertility over per capita labor. The reason is that a drop in fertility enlarges the weight of older workers in the labor force. If older workers are more productive than the average, then the drop in fertility increases the productivity of the labor force and, that would imply a reduction in the optimal retirement age. In this sense, the proposed model provides a measure that takes into account this last mechanism and determines the effect of a decrease in fertility over per capita labor: the average age of labor units, in which each age group is weighted by their contribution to the total amount of efficiency units of labor. If the average age of labor units is higher than the average age of total population, then a drop in fertility would imply an increase in the per capita labor and the optimal response would be a decrease in the retirement age. The average age of labor units can be also easily obtained from the data and may be used for political advise.

The paper does not only provide some qualitative theoretical results, it also uses empirical data to determine in which countries pronatalist policies are suitable to alleviate the negative consequences associated with ageing population and in which countries are not. In this sense, our paper greatly contributes to clarify the debate about the type of political tools needed to mitigate the consequences of ageing population. Using a wide sample of developed countries, and calculating several statistics described above, we find that a drop in fertility will increase the activity rate, the per capita efficiency unit of labor and will reduce the optimal retirement age in most of the countries in our sample. Thus, developed countries that are involved in policy reforms targeted to delaying the retirement age to overcome the drop in fertility, are not choosing the right solution.

The paper also analyzes the effect of economic growth on the optimal retirement age. To do it, economic growth is incorporated in the model through an exogenous technological change. The effect of a drop in the long run growth rate (a fall in the rate of technological change) is to increase the retirement age. Thus, the fall in the growth rate that developed countries have suffered during the last decades is another factor that contributes to understand the tendency in these countries to increase the retirement age.

Finally, the paper shows how the social planner allocation may be decentralized through an optimal social security system.

The paper that is most closely related to ours is the one written by Crettez and Le Maitre (2002). This contribution analyzes the social planer problem in an OLG model with the retirement age as an endogenous variable. The most empirically plausible case they considered was when the elasticity of substitution between old workers' labor and young worker's labor is higher than one. In this scenario, if the weight of a population group in the social planer's welfare function depends on its relative size, they found that the effect of fertility on the optimal retirement age is indeterminate. By contrast, our paper offers a more precise result: the effect of fertility on the optimal retirement age depends on the value of a well-defined measure, which may be easily obtained from the data in order to give a clear yardstick for political advice. Furthermore, the current set-up incorporates some elements which are very relevant for the retirement age debate, which are not possible to analyze in the model of Crettez and Le Maitre (2002), such as the effect of a higher life expectancy on the retirement age, the influence of children population to the dependency ratio and its consequences for retirement and the effect of the productivity profile along the life-cycle for optimal retirement.

Many papers in the literature have analyzed the effect of a public pension program on the individual's decision on retirement (see for example Kahn 1988; Fabel 1994). Recent literature has studied the decision of retirement from a political economy perspective (see Conde-Ruiz and Galasso 2003, 2004). Other papers have analyzed the decision of retirement from a social security reform environment (see Auerbach, Kotlikoff, Hagemann, and Nicoletti, 1989, Stock and Wise, 1990 and De Nardi, Imrohoroglu, and Sargent, 1999). Finally, another recent strand of the literature has focused on the impact of changes in fertility and longevity in determining the retirement age. Lacomba and Lagos (2006) analyzed the effects of population ageing on the retirement age using a life-cycle model. They found that the demographic effects of a decrease in the population growth rate may lead to a delay in the preferred retirement age, when the dependency ratio modifies the contribution rate (see also Bloom, Canning and Graham, 2004 and Fehr, Jokish and Kotlikoff, 2008). While all these previous contributions focus on how different social security schemes affect the individuals' decision of retirement, this paper steps back from the social security debate and examines the more general question of how different demographic changes affect the optimal retirement age from a social welfare viewpoint. This does not means that our conclusions are irrelevant to understand how demographic variables would affect a well design social security system, since we show how the optimal allocation may be decentralized through an optimal social security system.

Other branch of the literature analyzes different aspects of the economic consequences of an ageing population, for example Boadway et al. (1990 a, b), Marchand et al. (1990) and Meijdam and Verbon (1997). However, these theoretical papers provide no clear answers which can be upheld by empirical results. Indeed, the empirical literature on the fiscal and economic effects of fertility changes has produced controversial results. For example, Cutler et al. (1990), Guest and McDonald (2002), Guest (2006) and Heijdra and Ligthart (2006) found that declining fertility rates have a positive economic impact on future living standards, i.e., increases per capita consumption. While Berkel, Börsch-Supan, Ludwing and Winter (2004) find that a drop in the fertility rate would worsen the long-run pension finances. All these papers treat retirement age as an exogenous variable and thus, they cannot analyze how ageing population affects the optimal retirement age, which is the main goal of our paper.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 describes the social planner's problem. Section 4 analyzes the dynamic behavior of the economy. Section 5 presents and discusses the effects of an ageing population. Section 6 analyzes the effects of an increase in the growth rate. In section 7 we use empirical data to determine the effect of a drop in fertility over the labor force in a wide sample of developed countries. Section 8 shows how the optimal allocation may be decentralized through an optimal social security system. Finally, the last section concludes. The paper also includes a technical appendix.

## 2. The model

### 2.1. Demographic Dynamics:

Time is continuous and indexed by $t \in \mathbb{R}$. Population is composed of agents of different ages indexed by $a \in [0, \overline{a}]$, where $\overline{a}$ is the upper limit of duration of life. Live agents go through three stages: childhood, when $a \in [0, a_y)$; youth, when $a \in [a_y, a_o]$; and old age, when $a > a_o$, with $a_y < a_o$. Agents are only able to work during youth.

The probability of surviving at age $a$ is denoted by $s(a)$. To simplify, we assume that during childhood and youth, agents survive with probability one: $s(a) = 1, \forall a \leq$

$a_o$; while during old age the probability of being alive at age $a$ is equal to $s(a) = \psi(a;\xi), \forall a > a_o$, where $\psi : [a_o, \overline{a}] \times \mathbb{R}_+ \to [0,1]$, is a strictly decreasing function in its first argument and strictly increasing in its second argument when $a \in (a_o, \overline{a})$ and such that $\psi(a_o;\xi) = 1$ and $\psi(\overline{a};\xi) = 0$. Thus, agents survive for sure along childhood and youth, and are dead for sure when they reach the upper limit of duration of life $\overline{a}$. The probability of being alive during old age decreases with age and increases with the parameter $\xi$. Thus, an increase in $\xi$ implies an increase in life expectancy. In this sense, parameter $\xi$ could be considered as the health status or some biological ingredient that defines individuals' capacity to survive. To simplify, we will interpret an increase in $\xi$ as an increase in longevity.

The number of agents of age $a$ at time $t$ is denoted by $N(a,t)$ and is defined as follows:

$$N(a,t) = N(0, t-a)s(a) \tag{2.1}$$

That is, the number of agents of age $a$ at time $t$ is equal to the number of agents born "$a$" periods before, multiplied by the probability of being alive after "$a$" periods of being born.

We assume that births increase over time at a constant rate denoted by $n$:

$$\dot{N}(0,t) = nN(0,t) \tag{2.2}$$

Using equations (2.1) and (2.2) we get:

$$N(a,t) = N(0,t)s(a)e^{-na} \tag{2.3}$$

It follows from the above equation that the total population $N(t)$ can be calculated as:

$$N(t) = N(0,t)\int_0^{\overline{a}} s(a)e^{-na}da \tag{2.4}$$

It follows from the above equation and (2.2) that the total population increases over time at the constant rate $n$:

$$\dot{N}(t) = nN(t) \tag{2.5}$$

It follows from equations (2.3) and (2.4) that the fraction of agents of age $\widehat{a}$ in the whole population, $\mu(\widehat{a})$, is as follows:

$$\mu(\widehat{a}) = \frac{s(\widehat{a})e^{-n\widehat{a}}}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}, \quad \forall \widehat{a} \leq \overline{a} \tag{2.6}$$

where:

$$\int_0^{\overline{a}} \mu(a)da = \int_0^{\overline{a}} \frac{s(a)e^{-na}}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]}da = 1$$

We can rewrite the number of agents of age $a$ at time $t$ as a fraction of the total population at time $t$, that is,

$$N(a,t) = \mu(a)N(t) \tag{2.7}$$

**2.2. Technology:**

There exists a unique good that may be used either as a consumption good or as an investment good. There are two factors: labor $L$ and capital $K$. The amount of production is given by the Cobb-Douglas production function:

$$\Gamma(t)^{1-\alpha}K(t)^{\alpha}L(t)^{1-\alpha}$$

There is exogenous technological change:

$$\dot{\Gamma}(t) = \gamma\Gamma(t)$$

The accumulation of capital follows the conventional neoclassical law of motion:

$$\dot{K}(t) = I(t) - \delta K(t)$$

where $I(t)$ denotes gross investment and $\delta$ denotes depreciation rate. If we define $k(t) \equiv \frac{K(t)}{N(t)}$ as the per capita capital, we may rewrite the capital accumulation equation as follows:

$$\dot{k}(t) = i(t) - (\delta + n)k(t)$$

where $i(t) \equiv \frac{I(t)}{N(t)}$ denotes the per capita investment. Taking into account the fact that production is devoted to either investment or to consumption, the above equation may be rewritten as follows:

$$\dot{k}(t) = \Gamma(t)^{1-\alpha}k(t)^{\alpha}l(t)^{1-\alpha} - c(t) - (\delta + n)k(t) \tag{2.8}$$

where $l(t) \equiv \frac{L(t)}{N(t)}$ and $c(t) \equiv \frac{C(t)}{N(t)}$ denote respectively per capita labor (per capita efficiency units of labor) and per capita consumption.

**2.3. Welfare function and endowments:**

The welfare of the society consists of the sum of utility of their members, which depends on consumption and the effort of working:

$$\int_0^\infty \left[ \int_{\Lambda(t)} N(a,t) \left[ \ln(c(a,t)) - \phi(a) \right] da + \int_{[0,\overline{a}]-\Lambda(t)} N(a,t) \left[ \ln(c(a,t)) \right] da \right] \; e^{-\rho t} dt$$

where $c(a,t)$ denotes the consumption of the agents at age $a$ at time $t$, $\rho \in (0,n)$ denotes the subjective discount rate of the utility function. $\Lambda(t)$ denotes the correspondence which relates the time index $t$ with a measurable subset of $[a_y, a_o]$ which defines the ages at which agents work. Thus, if $a \in \Lambda(t)$, then agents at age $a$ work at time $t$. In other words, $\Lambda(t)$ defines the range of ages at which agents work. Agents either work or do not, but they cannot choose the amount of time devoted to work which is exogenous. Finally, $\phi(a)$ is the disutility derived from working, which is an increasing function of age $a$. Furthermore, $\phi(.)$ is continuous and differentiable of second order, convex and $\lim_{a \to a_o} \phi'(a) = +\infty$. This function could be interpreted as the subjective health cost of working. It is reasonable to assume that older workers have more health problems and thus, they have a larger cost. [4]

Using the assumptions about population behavior (see equations 2.5 and 2.7), we may rewrite the utility function as follows:

$$N(0) \left[ \int_0^\infty \left[ \int_0^{\overline{a}} \mu(a) \ln(c(a,t)) da - \int_{\Lambda_t} \mu(a) \phi(a) da \right] \; e^{-(\rho - n)t} dt \right]$$

As we assert above, only young agents may work, neither children or old age agents can work. We denote $h(a)$ the amount of labor that a young worker of age $a$ owns, where $h(a)$ is a function $h : [a_y, a_o] \to \mathbb{R}_{++}$, which is continuous and differentiable of second degree. We assume that $\frac{\phi(a)}{h(a)}$ is an increasing function. This assumption implies that the losses of working (the utility cost of working) grows faster with age than the gains of working (the endowment of labor). Thus, despite older workers may be the most productive, they suffer a higher subjective cost of working (this may be interpreted as older workers having more health problems). At

[4] Longitudinal data from the federal government's Health and Retirement Survey shows that the onset of major health problems frequently leads directly to withdrawal from the labor force (see Forman and Chen, 2008).

the end, the cost of working exceeds the gain of doing it. This implies that agents work at the beginning of their youth, and stop working (get retired) after a certain age, that we will call retirement age.

## 3. The Social Planner's Problem

In this section we present the social planner problem. Since the paper focuses on the characterization of the optimal retirement age from the social welfare point of view, we address our efforts to determine the optimal allocation that maximizes a welfare function, this is, the solution of a social planer problem. The aim of this approach is providing a welfare criterion which helps to clarify the goals that the public authorities should pursue in the ageing debate and consequently, which can contribute in the design of the most appropriated economic policies according to such objectives. In this regard, Section 8.2 shows how this optimal allocation can be decentralized through an optimal social security system. Thus, a suitable social security system may induce to households to retire at the optimal retirement age that would choose the social planner.

We consider a social planer who can choose the support of working ages and the allocations of consumption and capital throughout time. The social planner maximizes the welfare function choosing among feasible allocations of resources. The social planner's problem is as follows:

$$\max_{\{c(a,t)\}_{a=0}^{\overline{a}},\ \Lambda(t)} \int_0^\infty \left[\int_0^{\overline{a}} \mu(a)\ln(c(a,t))da - \int_{\Lambda(t)} \mu(a)\phi(a)da\right] e^{-(\rho-n)t}dt$$

$$\text{s.t.} \quad : \quad \dot{k}(t) = \Gamma(t)^{1-\alpha}k(t)^{\alpha}l(t)^{1-\alpha} - \int_0^{\overline{a}} \mu(a)c(a,t)da - (n+\delta)k(t) \tag{3.1}$$

$$l(t) = \int_{\Lambda(t)} h(a)\mu(a)da \tag{3.2}$$

where $l(t)$ is the per capita labor in the economy. Notice that the per capita labor supply is measured in efficiency units of labor and consequently is different from the per capita amount of workers (activity rate), $\int_{a_y}^{a_r(t)} \mu(a)da$.[5] The assumption that the instantaneous utility function derived from consumption $\ln(c)$ is strictly

[5]If the endowments of labor over ages are equal to 1, $l(t) = 1, \forall a$, then the per capita labor supply coincides with the per capita work force.

concave implies that consumption across agents will be identical for any optimal solution. The assumption that $\frac{\phi(a)}{h(a)}$ is an increasing function implies that at the optimal solution, younger agents work, while older workers do not. These older agents that do not work are called retirees. The retirement age, $a_r(t)$, is the age at which agent an ceases to work. That is, young agents work when their age is equal to or lower than $a_r(t)$ and they are retired when their age is larger than $a_r(t)$. Thus, the support of working ages is $\Lambda(t) = [a_y, a_r(t)]$. These features of the optimal solution allow us to rewrite the social planner optimization problem as follows:

$$\max_{c(t),\ a_r(t)} \int_0^{\infty} \left[ \ln(c(t)) - \int_{a_y}^{a_r(t)} \mu(a)\phi(a)da \right] e^{-(\rho-n)t} dt \tag{3.3}$$
$$\text{s.t.} \quad : \quad \dot{k}(t) = \Gamma(t)^{1-\alpha} k(t)^{\alpha} l(a_r(t))^{1-\alpha} - c(t) - (n+\delta)k(t)$$

where $l(a_r(t)) = \int_{a_y}^{a_r(t)} h(a)\mu(a)da$ is the amount of per capita labor supply in the economy (per capita labor units) as a function of the retirement age [6]. The labor supply is a increasing function of the retirement age: the larger the retirement age, the larger the amount of young agents that work and so, the larger the supply of labor.

The first order conditions and the transversality condition of the above problem are:

$$\begin{aligned}
\frac{1}{c(t)} e^{-(\rho-n)t} &= \lambda(t) \\
\phi(a_r(t)) e^{-(\rho-n)} &= \lambda(t)(1-\alpha)\Gamma(t)^{1-\alpha} \left( \frac{k(t)}{l(a_r(t))} \right)^{\alpha} h(a_r(t)) \\
\dot{\lambda}(t) &= -\lambda(t) \left[ \alpha\Gamma(t)^{1-\alpha} \left( \frac{k(t)}{l(a_r(t))} \right)^{\alpha-1} - (n+\delta) \right] \\
\lim_{t\to+\infty} \lambda(t)k(t) &= 0
\end{aligned}$$

where $\lambda(t)$ are the Lagrangian multipliers of the associated Hamiltonian with the social planner optimization problem (3.3). Using the above first order conditions we

[6] See Appendix 10.1 for a detailed description of the properties of labor supply function.

get:

$$\phi(a_r(t)) = \frac{1}{c(t)}\omega(t)h(a_r(t)) \tag{3.4}$$

$$\frac{\dot{c(t)}}{c(t)} = [r(t)-\rho] \tag{3.5}$$

$$\left[\frac{\phi'(a_r(t))a_r(t)}{\phi(a_r(t))} - \frac{h'(a_r(t))a_r(t)}{h(a_r(t))}\right]\frac{\dot{a}_r(t)}{a_r(t)} = -[r(t)-\rho] + \frac{\dot{\omega}(t)}{\omega(t)} \tag{3.6}$$

where

$$\omega(t) = (1-\alpha)\Gamma(t)^{1-\alpha}\left(\frac{k(t)}{l(a_r(t))}\right)^{\alpha} \tag{3.7}$$

$$r(t) = \alpha\Gamma(t)^{1-\alpha}\left(\frac{k(t)}{l(a_r(t))}\right)^{\alpha-1} - \delta \tag{3.8}$$

where $\omega(t)$ and $r(t)$ are respectively the return on labor and capital. The equation (3.4) describes the trade off between the gain in utility of not be working and the loss of labor income when an agent retires: if the social planner decides that the retirement age is $a_r(t)$, the individuals who retire at that age increase their utility when they do not work in $-\phi(a_r(t))$. However, there is a loss in labor income equal to $\omega(t)h(a_r(t))$, which reduces consumption and therefore, the utility derived from consumption in $\frac{1}{c(t)}\omega(t)h(a_r(t))$. The second equation, equation (3.5), is the typical optimal rule of intertemporal consumption choice: the choice between consumption in the present and consumption in the future depends on the degree at which consumption in the future is valuable, which depends on $\rho$, and the rate at which consumption in the present may be substituted by consumption in the future. Finally, equation (3.6) describes the trade off between retirement in the present and retirement in the future. The first term on the right side of this equation shows that if present generations retire later, this allows them to accumulate capital and this additional capital allows future generations to retire earlier (this is why the return on savings $r$ and the valuation of future generations $\rho$ affects such a choice). The second term on the right side is the growth rate of returns of labor $\frac{\dot{\omega}(t)}{\omega(t)}$, which represents the difference in the loss of labor income that present and future generations are going to suffer when they retire. If returns of labor increase, future generations are going to earn more than present ones, and in this sense it is better for present generations to retire earlier since the opportunity cost of retiring is not as high as in the future. Finally, the first term on left side of the equation takes into account the rate at which the retirement age of present generations may be substituted by the retirement age of future generations. This "elasticity of substitution" between retirement age in the present and in the future depends not only on the elasticity of the disutility of working with respect to age, $\frac{\phi'(a_r(t))a_r(t)}{\phi(a_r(t))}$, but also on the elasticity of the productivity with respect to age. [7]

Finally, the transversality condition may be written in the following two alternative ways:

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{c(t)}e^{-(\rho-n)t}k(t) &= 0\\ \lim_{t\to+\infty}\frac{\phi(a_r(t))}{\omega(t)h(a_r(t))}e^{-(\rho-n)t}k(t) &= 0\end{aligned}$$

and the interpretation is as usual: nothing should be saved in the last period unless it is costless to do so.

## 4. Dynamic Behavior

It follows from the optimal solution or Pareto Optimal Allocation definition that the dynamic behavior of the economy may be described by the following dynamic system:

$$\dot{a}_r(t) = \frac{\rho+(\delta+\gamma)\,(1\text{-}\alpha)-\alpha n-\alpha(1\text{-}\alpha)\frac{h(a_r(t))}{\phi(a_r(t))l(a_r(t))}\left(\frac{l(a_r(t))}{\widetilde{k}(t)}\right)^{1-\alpha}}{\left[\frac{\phi'(a_r(t))}{\phi(a_r(t))}-\frac{h'(a_r(t))}{h(a_r(t))}+\alpha\frac{\mu(a_r(t))h(a_r(t))}{l(a_r(t))}\right]} \tag{4.1}$$

$$\dot{\widetilde{k}}(t) = \widetilde{k}(t)^{\alpha}l(a_r(t))^{1-\alpha}\left[1-\frac{(1\text{-}\alpha)h(a_r(t))}{\phi(a_r(t))l(a_r(t))}\right]-(\delta+n+\gamma)\widetilde{k}(t) \tag{4.2}$$

$$\lim_{t\to+\infty}\frac{\phi(a_r(t))}{(1-\alpha)}\left(\frac{\widetilde{k}(t)}{l(a_r(t))}\right)^{1-\alpha}e^{-(\rho-n)t} = 0$$

where $\widetilde{k}(t)\equiv\frac{k(t)}{\Gamma(t)}$. Figure 4.1 describes the dynamic behavior of the system defined above. We observe that the dynamic behavior is the typical saddle point dynamic.

[7] Notice that one of the conditions (3.5) and (3.6) is redundant, since (3.4) and (3.5) imply (3.6); and (3.4) and (3.6) imply (3.5).

So, there is a unique solution path that converges to the steady state, $(\widetilde{k}^{ss}, a_r^{ss})$ [8]. We can also observe that along the solution path, the retirement age decreases with capital. This is due to the wealth effect, since early retirement is a normal good: when wealth increases agents retire earlier.

Let's define $\widetilde{c}(t) \equiv \frac{c(t)}{\Gamma(t)}$. We define steady state as a solution of the social planer problem in which the variables $\widetilde{k}(t)$, $\widetilde{c}(t)$, $a_r(t)$ stay constant over time. The value of these variables at the steady state will be denoted as follows: $\widetilde{k}^{ss}$, $\widetilde{c}^{ss}$, $a_r^{ss}$. From now on, we will use this notation for all variables valued in the steady state.

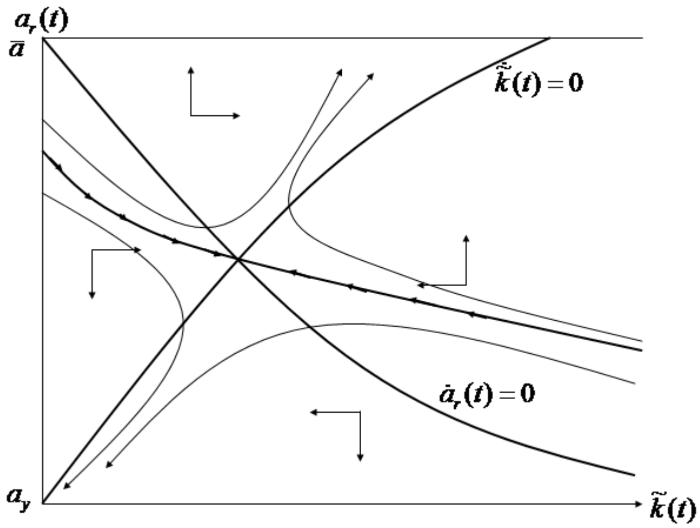


Figure 4.1: **Dynamic Behavior**

## 5. The effect of an ageing work force

In this section, we study the effect of the aging of the work force. There are two possible sources of aging: the increase in life expectancy and the drop in the fertility rate. While, at the first glance both changes looks like to generate the same effects, a deeper analysis reveals that this intuition may not be true. In fact, we show that these two factors may have, under quite empirically plausible assumptions, exactly

[8]See Appendix 10.2 for details.

the opposite effect on the labor supply and the economy: an increase in the life expectancy always increases the optimal retirement age at the steady state; while a drop in the fertility rate reduces the optimal retirement age at the steady state under some conditions that are precisely specified later on.

In this section, we omit the time label ($t$), whenever this does not cause any confusion.

### 5.1. The effect of an increase in life expectancy:

**Proposition 5.1.** *At the steady state,* $\frac{\partial a_r^{ss}}{\partial \xi} > 0$ $\frac{\partial \widetilde{c}^{ss}}{\partial \xi} < 0$ *and* $\frac{\partial \widetilde{k}^{ss}}{\partial \xi} < 0$.

An increase in parameter $\xi$ implies an increase in survival probabilities and therefore, an increase in life expectancy. Notice that the increase in parameter $\xi$ only affects the dynamic system (4.1)-(4.2) through the per capita labor supply. The labor supply is a decreasing function of $\xi$ since an increase in life expectancy increases the old population and therefore, increases the dependency rate and reduces the activity rate and labor supply[9]. Thus, an increase in life expectancy works as a negative wealth effect: it reduces the per capita consumption (and the per capita capital) and it increases the retirement age.

Figure 5.1 shows the effect of an increase of $\xi$ on the steady state and the transitory dynamics. The increase of life expectancy reduces per capita labor supply, and this has a negative effect on per capita resources of households, reducing consumption and increasing the retirement age.

### 5.2. The effect of an increase in the fertility rate:

An increase in the fertility rate affects the retirement age through four mechanisms. The first three mechanisms are related to the effect of fertility on per capita labor supply. The offsetting nature of them makes the net effect of labor supply ambiguous at the first sight. These three mechanisms are the following: *i*) an increase in the fertility rate reduces the weight of old agents, increasing the activity rate. *ii*) It also increases the weight of children, reducing the activity rate. Thus, the effect on the dependency ratio seems to be ambiguous. *iii*) Furthermore, owing to the fact

[9]See Appendix 10.1.

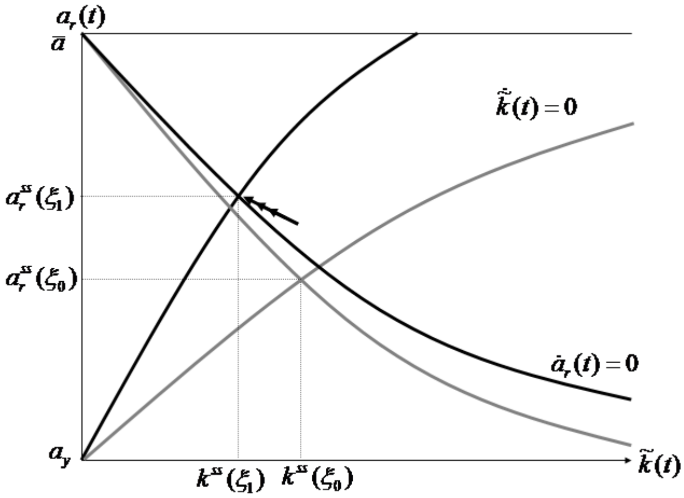


Figure 5.1: **The effect of an increase in life expectancy**

that workers' labor productivity may increase with experience (age), an increase in the fertility rate may reduce the weight of more experienced and productive groups inside the labor market, reducing labor productivity and therefore the per capita labor supply. Finally, the forth mechanism is not related to the labor supply but to the amount of investment required to keep the per capita capital constant. When the fertility rate increases, a larger amount of investment is required in order to keep the per capita capital constant. Thus, this mechanism acts as an increase in the depreciation rate, involving a negative "wealth effect".

The first two mechanisms mentioned above are related to the dependency ratio and the activity rate: the effect of an increase in fertility on these two ratios seems to be ambiguous since the rise of fertility decreases the portion of older population, which reduces the dependency ratio (increases the activity rate), but increases the portion of younger population which increases the dependency ratio (decreases the activity rate). Let's define activity rate and dependency ratio as follows:

$$AR = \frac{\int_{a_y}^{a_r} \mu(a)da}{\int_0^{\overline{a}} \mu(a)da} = \int_{a_y}^{a_r} \mu(a)da \tag{5.1}$$

$$DR = \frac{\int_0^{a_y} \mu(a)da + \int_{a_r}^{\overline{a}} \mu(a)da}{\int_{a_y}^{a_r} \mu(a)da} = \frac{\int_0^{\overline{a}} \mu(a)da - \int_{a_y}^{a_r} \mu(a)da}{\int_{a_y}^{a_r} \mu(a)da} \tag{5.2}$$

That is, the activity rate is equal to the number of workers divided by total population and dependency ratio is equal to children plus retired population divided by active population. The dependency ratio is a decreasing function of the activity rate:

$$DR = \frac{\int_0^{\overline{a}} \mu(a)da - \int_{a_y}^{a_r} \mu(a)da}{\int_{a_y}^{a_r} \mu(a)da} = \frac{\frac{\int_0^{\overline{a}} \mu(a)da}{\int_0^{\overline{a}} \mu(a)da} - \frac{\int_{a_y}^{a_r} \mu(a)da}{\int_0^{\overline{a}} \mu(a)da}}{\frac{\int_{a_y}^{a_r} \mu(a)da}{\int_0^{\overline{a}} \mu(a)da}} = \frac{1-AR}{AR} = \frac{1}{AR} - 1$$

We define the average age of population $E[a]$ and the average age of workers $E[a/[a_y, a_r]]$ as follows:

$$E[a] \equiv \frac{\int_0^{\overline{a}} a\ \mu(a)\ da.}{\int_0^{\overline{a}} \mu(a)da} = \int_0^{\overline{a}} a\ \mu(a)\ da.$$

$$E[a/[a_y, a_r]] \equiv \frac{\int_{a_y}^{a_r} a\ \mu(a)\ da.}{\int_{a_y}^{a_r} \mu(a)\ da}$$

**Proposition 5.2.** $\frac{\partial AR}{\partial n} < 0$ *(*$\frac{\partial DR}{\partial n} > 0$*) if and only if* $E[a/[a_y, a_r]] > E[a]$.

The meaning of the previous proposition is that an increase in fertility rate will reduce the activity rate and will increase the dependency ratio when the average age of workers is larger than the average age of total population. Otherwise, the effect of the increase on fertility on activity rate and dependency ratio would be just the opposite. Thus, this proposition establishes a very clear yardstick to determine the effect of an increase of fertility rate on the activity rate and dependency ratio. Furthermore, this yardstick may be easily obtained from the data as we will have the opportunity of verifying in section 7. In such section this criterium is used to discuss the effect of a drop in fertility over the activity rate and dependency ratio for a wide sample of developed countries.

To analyze the effect of an increase in the fertility rate on the determination of the optimal retirement age, what really matters is how the per capita labor supply is affected, that is, the impact on the per capita amount of efficiency units of labor. Therefore, in order to analyze the effect of an increase in the fertility rate on per capita labor supply, we define the labor supply contribution density function as follows:

$$\nu(a; a_r) = \frac{h(a)\mu(a)}{\int_{a_y}^{a_r} h(a)\mu(a)da}$$

this density function gives the weight that each type-$a$ agent has on the labor supply. Note that the more productive an agent is, the higher her weight in the labor supply. We define the average age of labor units as the average age of workers weighted by their contribution to the labor units, that is, using the labor supply contribution density function:

$$E_\nu \left[a/ \left[a_y, a_r\right]\right] = \int_{a_y}^{a_r} a\, \nu(a; a_r)da$$

**Proposition 5.3.** $\frac{\partial l(a_r)}{\partial n} < 0$ *if and only if* $E_\nu \left[a/ \left[a_y, a_r\right]\right] > E\left[a\right]$.

Proposition 5.3 establishes that an increase in the fertility rate reduces the per capita labor supply if and only if the average age of labor units is higher than the average age of total population. Thus, this proposition provides a clear and empirically observable condition to determine the effect of an increase in the fertility rate over the per capita labor supply: if the average age of labor units is larger than the average age of total population a rise in fertility reduces the per capita labor supply. As we observed in Table 7.1, for all developed countries in the sample, except Italy, the average age of labor units is higher than the average age of workers. This implies that an increase in the fertility rate in all these countries would reduce the per capita labor supply.

The necessary condition in proposition 5.3 implies the following sufficient one:

**Corollary 5.4.** *If* $h(a_r)$ *is a cuasi-concave function,* $E\left[a/ \left[a_y, a_r\right]\right] > E\left[a\right]$ *and* $h(a_r) > E\left[h(a)/ \left[a_y, a_r\right]\right]$, *then* $\frac{\partial l(a_r)}{\partial n} < 0$.

That is, if the productivity-age profile displays a hump shape ($h(a_r)$ is cuasi-concave), the average age of the workers is greater than the average age of the total population, and the number of efficiency units of labor of agents at the retirement age is larger than the average number of efficiency units of labor of the work force, then an increase in the fertility rate reduces per capita labor supply. In this proposition, we can see two different effects: first, when $E\left[a/\left[a_y, a_r\right]\right] > E\left[a\right]$, that is, when the average age of workers is greater than the average age of total population, an increase in the fertility rate reduces the activity rate, as we showed in proposition 5.2. Second, when $h(a_r) > E\left[h(a)/\left[a_y, a_r\right]\right]$, that is, when the number of efficiency units of labor of agents at retirement age is larger than the average of the age profile then, an increase in the fertility rate reduces the weight of the older and more experienced and productive workers, which implies a drop in the average efficiency units of labor in the economy. As we pointed out above, table 7.1 showed that in most countries, the average age of the workers is higher than the average age of the population. Moreover, the abundant empirical evidence in favor of a hump-shaped wage-productivity profile in developed countries supports the fact that productivity at retirement age is greater than the average productivity throughout the life cycle.

Thus, we can conclude that the results in proposition 5.3, together with table 7.1, imply that the most relevant case from the empirical point of view is the one in which an increase in fertility rate reduces the per capita labor supply. For this reason, we will focus the analysis in this case.

**Proposition 5.5.** *At the steady state, if* $E_\nu\left[a/\left[a_y, a_r\right]\right] > E\left[a\right]$ *then* $\frac{\partial a_r^{ss}}{\partial n} > 0$ *and* $\frac{\partial c^{ss}}{\partial n} < 0$.[10]

Summarizing, according to our results, a drop in the fertility rate increases the weight of older agents in the population, as the increase in life expectancy does. However, we have proved that these two sources of ageing population may have just

[10]The condition $E_\nu\left[a/\left[a_y, a_r\right]\right] > E\left[a\right]$ is a sufficient condition. The necessary and sufficient condition for this proposition would be as follows:

$$E_\nu\left[a/\left[a_y, a_r\right]\right] > E\left[a\right] - \frac{\alpha\left[1 + (\delta + \rho + \gamma)\right]}{\left[(1-\alpha)\left(\delta + \rho + \gamma\right) + \alpha(\rho - n)\right]\left[1 + \alpha\left(\delta + \rho + \gamma\right)\right]}$$

Thus, even in the case in which an increase in the fertility rate increases the per capital labor supply, which occurs when $E_\nu\left[a/\left[a_y, a_r\right]\right] > E\left[a\right]$, this do not guarantee that an increase in the fertility rate will reduce the retirement age, since the fourth mechanism explained above may generate the opposite effect.

the opposite effect on the optimal retirement age: the increase of life expectancy always rises the optimal retirement age, while the drop of fertility have just the opposite effect under the condition specified in the proposition 5.5, which is hold for most countries (see table 1.1). This drop in fertility involves four counterweighted mechanisms that affect the optimal retirement age: *i*) the per capita number of children falls, increasing the per capita work force; *ii*) the per capita number of retirees rises, reducing the per capita work force; *iii*) the older and generally more productive workers gain weight in the work force, increasing potentially the number of efficient units of labor per worker and, consequently, the per capita labor supply and; *iv*) Finally, the amount of investment needed to keep the amount of per capita capital constant drops. The three first mechanisms determine the sign of a drop of fertility in the per capita labor supply. We have shown that proposition 5.3 and table 1.1 implies that the most relevant case from the empirical point of view is that a drop of fertility involves a rise in the per capita labor supply, which is the case we concentrate in proposition 5.5. This increase of per capita labor supply and the reduction of the investment required to keep the per capita capital constant (mechanism four) act as a positive wealth effects, reducing the optimal retirement age and an increasing the amount of per capita consumption and capital at the steady state. Figure 5.2 shows the new steady state and the dynamics due to a reduction in the fertility rate.

## 6. The effect of a technological improvement

**Proposition 6.1.** $\frac{\partial a_r^{ss}}{\partial \gamma} < 0$ *if and only if* $\rho - n < \frac{1-\alpha}{\alpha}$ *(*$\frac{\partial a_r^{ss}}{\partial \gamma} > 0$ *if and only if* $\rho - n > \frac{1-\alpha}{\alpha}$*).*

An increase in the growth rate of the technological change $\gamma$ involves three offsetting mechanisms that affect the optimal retirement age (see equation 3.6): *i*) the rate at which wages grow increases which involves a drop in the present labor supply and in the retirement age due to a substitution effect: the opportunity cost of retiring today in terms of retiring in the future falls due to the higher wages in the future; *ii*) there is another substitution effect with just the opposite effect: there is an increase in the return of savings, which makes more “expensive” to retire today

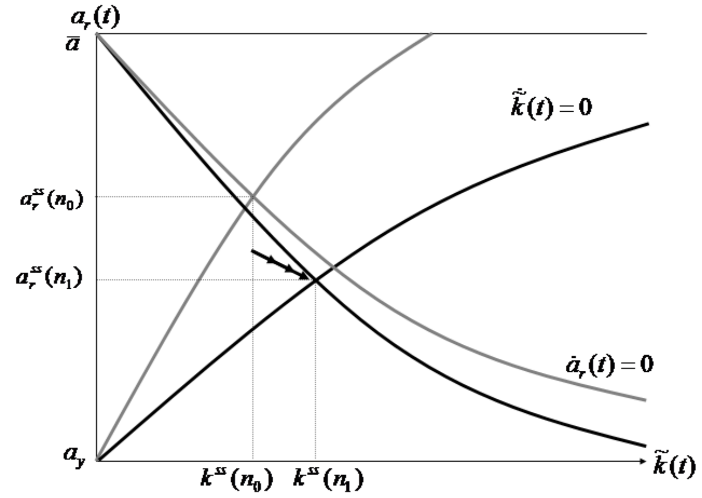


Figure 5.2: **The effect of a decrease in the fertility rate**

with respect to retire in the future and; *iii*) finally, there is a wealth effect: a higher growth of wages, implies a higher present value of labor income, generating a positive wealth effect, which involve a lower retirement age. These three mechanisms make the effect of an increase in the technological speed $\gamma$ ambiguous at the first sight. However, as the above proposition asserts, the most plausible case from the empirical point of view is that the increase in the technological speed generates a drop in the retirement age[11].

## 7. Empirical application: evaluating the effect of a drop in fertility in developed countries

We have shown that our model has two predictions: *i*) the first prediction is that if the average age of the workers is greater than the average age of the population, then a drop in fertility will imply an increase in the activity rate; *ii*) the second prediction is that if the average age of efficiency unit of labor is greater than the average age of the population, then a drop in fertility will imply an increase in the

[11] In standard calibrations the discount rate of the utility function $\rho$ is usually around 0.04, while the labor share $(1-\alpha)$ is around 2/3, which implies that $(1-\alpha)/\alpha$ is around 2. Obviously, the necessary and sufficient condition of the above proposition holds with these parameters.

per capita labor and the optimal retirement age. These predictions involve three statistics that can be easily obtained from the data: average age of the workers, the average age of the population and the average age of efficiency unit of labor. Thus, If we calculate these statistics for a particular country, then we can make predictions about the effect of a drop in fertility in such country. The goal of this section is to use a sample of developed countries to determine in which of these a drop in fertility will reduce the retirement age and will mitigate the effect of ageing population in the labor supply. In this way, we can give some guidance about the effectiveness of pronatalist policies in these countries.

Table 7.1: Average age population and average age of work force

| Countries[a] | Av. age of population | Av. age of work force | Av. age of labor supply |
|---|---|---|---|
| Australia | 37.07 | 39.60 | 41.76 |
| Canada | 38.65 | 40.41 | 42.70 |
| Denmark | 39.02 | 41.30 | 42.77 |
| Finland | 39.97 | 41.39 | 43.54 |
| France | 38.83 | 39.75 | 41.50 |
| Germany | 41.74 | 41.01 | 42.67 |
| Italy | 42.18 | 39.76 | 41.30 |
| Japan | 42.40 | 41.80 | 43.36 |
| Netherlands | 38.70 | 39.89 | 40.45 |
| Spain | 39.96 | 38.61 | 41.45 |
| Sweden | 40.53 | 42.03 | 44.04 |
| United Kingdom | 39.00 | 40.40 | 41.27 |
| United States | 36.40 | 40.29 | 42.88 |

[a]Source: Data on population weights were obtained from the US Census Bureau. Working population is measured as the active population aged from 20 to 64 years old. Data on activity rates were obtained from the OECD database. Relative weights of age workers groups used to obtain the average age of the labor supply have been calculated using the relative hourly wages by age group. Data on the wage profiles were obtained from OECD report (1998).

We now proceed to calculate empirically the measures we have described before. The first and the second columns in Table 7.1 show the average age of the population and the average age of the workers, respectively, for a wide sample of developed countries in 2005. In most of these countries, the average age of the workers is greater than the average age of the population.[12] In this case, our model predicts that a drop in fertility will increase the activity rate and will reduce the optimal retirement age. Thus, according to the results of the paper, the effect of a drop in fertility over the activity rate would be positive in these countries. The exceptions are Germany, Italy, Japan and Spain. The third column in Table 1 shows the average age of labor units. The average age of labor units is higher than the average age of workers in all the countries of the sample, implying that the contribution of older to the labor supply is greater than others; i.e., older workers are the most productive.[13] More interestingly is the fact that the average age of labor units is higher than the average age of population. In this case, our model predicts that a drop in fertility will increase the amount of efficiency units of labor and will reduce the optimal retirement age. According to this result, a drop in fertility would produce an increase in per capita labor and the optimal response to it should be a reduction in the retirement age. Thus, developed countries [14] in our sample that are involved in policy reforms targeted to delaying the retirement age to overcome the drop in fertility, are not choosing the right solution.

## 8. How to decentralize the Pareto Optimal Allocation:

There are at least two ways of decentralizing the efficient allocation: *i*) considering that there exists perfect inter-generational altruism and, *ii*) considering that there exists an optimal social security system.

### 8.1. Perfect Inter-generational Altruism

The Pareto Optimal Allocation resulting from the solution of the social planner's problem can be also obtained as the result of a decentralized economic equilibrium

[12] The fact that the average age of the work force is higher than the average age of population is well documented in the literature (see for example, Börsch-Supan, 2002 and Prskawetz, Fent and Guest, 2008).

[13] There exists a great deal of empirical evidence in favor of a hump-shaped wage-productivity profile in developed countries. The pattern implies that the wage-productivity for older workers is greater than the average of the whole age profile (aee for example Kydland, 2004: Report of the National Equality Panel, 2010; Blanchet et al., 2005; Luong and Hérbert, 2009). Thus, the average age of labor units is larger than the average age of the working population.

[14] The exception is Italy.

with a representative household. Such representative household should have the following characteristics: *i*) it is composed by members of different ages in the same proportion as the average of the economy, *ii*) it has an amount of assets equal to the average in the economy and, *iii*) it has the same utility function as the welfare function defined in equation (3.3). If households in the economy are representatives households as we just described them, then, the equilibrium allocation will be the same that the social planner allocation. A representative household face the following optimization problem:

$$\begin{aligned} \max_{c(t),\ a_r(t)} \int_0^\infty \left[ \ln(c(t)) - \int_{a_y}^{a_r(t)} \mu(a)\phi(a)da \right] e^{-(\rho-n)t} dt \qquad (8.1) \\ \text{s.t.: } \dot{b}(t) = w(t)l(a_r(t)) + r(t)b(t) - c(t) - nb(t) \end{aligned}$$

where $b(t)$ denotes the the household's per capita amount of assets and $w(t)$ and $r(t)$ denote the prices of respectively the labor and the capital. The firm's maximization problem would be as follows:

$$\max_{l(t),\ k(t)} \Gamma(t)^{1-\alpha} k(t)^\alpha l(t)^{1-\alpha} - w(t)l(t) - (\delta + r(t))k(t) \qquad (8.2)$$

The labor and capital market equilibrium conditions would be as follows:

$$l(t) = l\left(a_r(t)\right) \qquad (8.3)$$

$$k(t) = b(t) \qquad (8.4)$$

where $l(t)$ and $k(t)$ are respectively the per capita demand of labor and capital by firms (the solution of the maximization problem of the firm, 8.2) and $l\left(a_r(t)\right)$ and $b(t)$ are respectively the per capita supply of labor and capital by households (the solution of the household's maximization problem, 8.1). Thus, equation (8.3) is the labor market clearing condition (the labor demand and supply should be equal at equilibrium) and equation (8.4) is the capital market clearing condition. The first order conditions of the household's problem (8.1) and the firm's problem (8.2), together with the labor and capital market clearing conditions, (8.3) and (8.4) imply that:

$$\phi(a_r(t)) = \frac{1}{c(t)}w(t)h(a_r(t)) \tag{8.5}$$

$$\frac{\dot{c(t)}}{c(t)} = [r(t) - \rho] \tag{8.6}$$

$$\left[\frac{\phi'(a_r(t))a_r(t)}{\phi(a_r(t))} - \frac{h'(a_r(t))a_r(t)}{h(a_r(t))}\right]\frac{\dot{a}_r(t)}{a_r(t)} = -[r(t) - \rho] + \frac{\dot{w}(t)}{w(t)} \tag{8.7}$$

$$w(t) = (1-\alpha)\Gamma(t)^{1-\alpha}\left(\frac{k(t)}{l(a_r(t))}\right)^{\alpha} \tag{8.8}$$

$$r(t) = \alpha\Gamma(t)^{1-\alpha}\left(\frac{k(t)}{l(a_r(t))}\right)^{\alpha-1} - \delta \tag{8.9}$$

Comparing these conditions with the ones resulting from the Pareto Optimal Allocation (see equations 3.4 to 3.8), it is easy to see that they coincide. Therefore, it is straightforward to prove that the Pareto Optimal Allocation coincides with the equilibrium one.

We will denote from now on the optimal allocation and the prices of the equilibrium that decentralize the Pareto Optimal Allocation with a star, this is, $c^*(t)$, $a_r^*(t)$, $w^*(t)$, $r^*(t)$, etc.

### 8.2. Optimal Social Security

Even in the case that agents would not be altruistic and they only would care about their own utility, it is possible to decentralize the Pareto Optimal Allocation through a social security system with the suitable tax-transfer scheme. We will call *optimal social security system* to such tax-transfer scheme that decentralizes the efficient allocation.

In order to define the optimal social security system, we need first to know what are the main features of the Pareto Optimal Allocation we want to decentralize. In this regard, if we look at the social planner's allocation we notice that optimal allocation of consumption implies that at any period, individuals of different ages have the same level of consumption. Since the social planner only considers individuals that are alive in each period, the social planner's decision of how to allocate resources between current and future consumption is not affected by the survival probability of a particular individual. This is reflected in the social planer's Euler equation which not includes the survival probability. However, in the market, when individuals only care about themselves, the survival probability affects their decisions about present and future consumption (reflected in the Euler Equation). The fact that they may die in the next period increases the discount of the future. As a result, the lower the probability of surviving, the higher the incentive to consume more in the present than in the future and so, the lower the amount of current savings. Thus, if the return of saving is the same for all individuals, it would be impossible to get the same allocation of the social planner in which the discount rate is not affected by the surviving probability. Consequently, we would have to modify the setting of the model in order to decentralize the Pareto Optimal Allocation. In particular, we would have to design a mechanism which encourages agents to increase their savings above the level derived from the current setting. In other words, we would have to introduce some mechanism in the economy which allows individuals to insurance from the risk of staying alive and thus, generating an incentive to increase savings and the level of consumption in the future. A simple way to do so would be to consider the existence of financial intermediaries which insure agents of the risk of staying alive offering assets differentiated by the age of the agents. Another alternative way would be considering complete asset markets. In this section, for pure expositional reasons, we will focus on the case of financial intermediaries. Despite of both forms of decentralization offering the same results, we consider that this way of decentralization is more intuitive than the decentralization with complete asset markets.

We now proceed to describe how financial intermediaries operate in the economy and afterwards, we will describe how the optimal social security system should be designed to decentralize the Pareto Optimal Allocation in this economy with those financial intermediaries.

**Financial Intermediaries:** We consider that financial intermediaries offer an asset whose return is contingent on the survival of the buyer. This is, if an agent survives to the next period she gets a return equal to $r(a(t), t)$, however, if the agent does not survive, financial intermediaries will not give any return to their heirs and will not refund them the principal. In some sense, this contract would be an opposite contract to life insurance: the agent gets the return if she survives, otherwise she (her heirs) would not receive anything.

Financial intermediaries also may buy assets, this is, agents may get loans from financial intermediaries. In this case, if the agent survives, she has to pay the principal plus the interest according with an interest rate which depends on the agent's age, $r(a(t),t)$, and if she does not survive, the bank cannot charge the debt to the assets of holder's heirs. The interest rate would depend on the agent's age because age determines the survival probability and, consequently, the probability that financial intermediaries can get back her loan. The arbitrage condition for this type of contingent asset would be as follows in discrete time:

$$(1+r_{t+1}(a))\frac{s\,(a+1)}{s\,(a)}=(1+r_{t+1})$$

where $r_{t+1}(a)$ is the return of the asset that an agent of age $a$ at period $t$ would obtain if she survives at age $a+1$; $\frac{s(a+1)}{s(a)}$ is the probability that an agent of age $a$ survives at age $a+1$ (or the probability of surviving at age $a+1$ conditioned to have been survived at age $a$). The continuous version of the above arbitrage condition would be as follows:

$$r(a,t)+\frac{\dot{s}\,(a)}{s\,(a)}=r(t) \tag{8.10}$$

where $\dot{s}\,(a)\leq 0$ is the slope at which the probability of being alive at age $a$ decreases; $\frac{\dot{s}(a)}{s(a)}$ is the probability of surviving to age $a$ conditional to stay alive at age $a$ and so; $-\frac{\dot{s}(a)}{s(a)}$ is the probability rate of not surviving to age $a$ conditional to stay alive in period $a$. Thus, the above arbitrage condition means that the contingent return on the asset offered by financial intermediaries in case of the agent of age $a$ survives, $r(a,t)$, minus the probability that she does not survive in the following period, $-\frac{\dot{s}(a)}{s(a)}$, obtaining nothing, should be equal to the return of a non contingent asset $r(t)$.

In the case that an old agent borrows from a financial intermediary, the arbitrage condition (8.10) still holds. In this case, the condition means that if the agent dies then, financial intermediaries cannot recover the loan. Thus, financial intermediaries impose an interest rate higher than the return of the non contingent asset to compensate the expected loses in the event of the borrower's death.

**Social Security System:** The social security system consists in a tax-transfer scheme that evolves with the age of the agent and along time. The tax that an agent with age $a$ pays at time $t$, will be denoted by $T(a,t)$. This tax may be negative, which means that the tax is a positive transfer. Transfers are denoted by $tr(a,t)$, being the transfer equal to the tax with the opposite sign: $tr(a,t) \equiv -T(a,t)$. Note that we are introducing not distortionary lump-sum taxes. Nevertheless, it is easy to introduce distortionary taxes, like a tax on labor income, without changing the results, as example 8.2 shows below.

In order to reproduce the same demographic structure as in subsection 2.1, we consider that each agent gives birth to $e^{na_y}$ children at age $a_y$, that is, the age at which individuals start working.[15] For expositional motives, we assume that the length of the youth stage (the range of ages at which agents can work) is larger than the childhood period, $a_y < a_o - a_y$ and, we also consider the case in which agents retire after their children start working, $2a_y < a_r$, [16], this is, when agents become grandparents. Furthermore, we assume that agents only care about their own consumption and leisure, and about their children's consumption when those are infants[17]. Thus, an agent which is born at period $t_b$, would face the following

[15]The age at which agents have children is irrelevant for the results. We have assumed that agents have children at the age at which they start working because is intuitive and simplifies the exposition, but many other alternative assumptions are compatible with the results.

[16]A sufficient condition to guaranty this statement is assuming that the subjective cost of working is zero, $\phi(a) = 0$, if $a \in [a_y, 2a_y]$.

[17]An alternative is that agents are not altruistic at all, that is, they do not care about they own children even when they are in the childhood. This assumption would not be incompatible with the decentralization of the Pareto Optimal Allocation. In this case, agents would need to incur in debts to finance their own consumption during childhood. We consider that the adoption of the assumption that agents care about their children during childhood is much more realistic, but it is not essential for the existence of an optimal social security system.

optimization problem when she starts working at age $a_y$:

$$\max_{[c(t)]_{t_b+a_y}^{t_b+\overline{a}},[c_{ch}(t)]_{t_b+a_y}^{t_b+2a_y},a_r} \int_{t_b+a_y}^{t_b+2a_y} \left[\ln(c(t)) + e^{na_y}\ln(c_{ch}(t))\right] e^{-\rho(t-(t_b+a_y))}dt +$$
$$\int_{t_b+2a_y}^{t_b+a_o} \ln(c(t))e^{-\rho(t-(t_b+a_y))}dt + \int_{t_b+a_o}^{t_b+\overline{a}} \ln(c(t))s(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt -$$
$$\int_{t_b+a_y}^{t_b+a_r} \phi(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt$$
$$\text{s.t.:} \tag{8.11}$$
$$\dot{b}(t) =$$
$$\begin{cases} w(t)h(t\text{-}t_b)\text{-}T(t\text{-}t_b,t)\text{+}r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t)\text{-}e^{na_y}c_{ch}(t) & \text{if } t \in [t_b\text{+}a_y, t_b\text{+}2a_y] \\ w(t)h(t\text{-}t_b)\text{-}T(t\text{-}t_b,t)\text{+}r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t) & \text{if } t \in [t_b\text{+}2a_y, t_b\text{+}a_r] \\ tr(t\text{-}t_b,t) + r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t) & \text{if } t \in [t_b\text{+}a_r, \overline{a}] \end{cases}$$
$$b(t_b + a_y) = 0;\ b(t_b + \overline{a}) \geq 0$$

where $c_{ch}$ denotes the consumption of each child. Thus, the life cycle of the agent may be subdivided in several stages:

*i*) Childhood: This stage ranges from her birth at $t_b$ until she starts working, at age $a_y$ at time $t_b + a_y$. In this stage the agent does not take any decision, her consumption is provided by her progenitor;

*ii*) Youth with children: This stage ranges from the moment she starts working and she gives birth to her children, at age $a_y$ at time $t_b + a_y$, until the moment in which her children start working, at age $a_y$, this is, when she is $2a_y$ years old at time $t_b + 2a_y$. In this stage the agent works and provide consumptions to her children.

*iii*) Youth without children: This stage ranges from the moment in which agent's children start working, at age $a_y$, when the agent is $2a_y$ years old at time $t_b + 2a_y$ until she retires, at age $a_r$ at time $t_b + a_r$. In this stage the agent works and only cares about her future profile of consumption.

*iv*) Youth retired: This stage ranges from the moment she retires, $t_b + a_r$, at age $a_r$, until the moment in which her probability of dying becomes positive, at age $a_o$ at time $t_b + a_o$. In this stage the agent does not work and finances her consumption with her assets and transfers.

*v*) Old age: This stage ranges from the moment in which her probability of dying becomes positive, at age $a_o$ at time $t_b + a_o$, until the moment she dies, which is uncertain and depends on the surviving probability. In this stage, the agent does not work and finances her consumption with transfers and assets which are contingent on her age and give her an extra return when she survives. Since she does not care about her descendants, she consumes her assets and her descendants do not get any bequest.

The necessary conditions of the optimization problem (8.11) are the following:

$$\begin{aligned}
\phi(a_r) &= \frac{1}{c(t_b+a_r)} w(t_b+a_r)h(a_r) && (8.12)\\
\frac{\dot{c}(t)}{c(t)} &= \left[r(a,t)-\rho+\frac{\dot{s}(a)}{s(a)}\right]\forall t && (8.13)\\
c_{ch}(t) &= c(t) \qquad t\in[t_b+a_y, t_b+2a_y] && (8.14)\\
b(t_b+\overline{a}) &= 0 && (8.15)
\end{aligned}$$

Note that the first order condition about retirement age (equation 8.12) is the same as in the optimal solution (see equation 8.5). Thus, if the resulting allocation of consumption and the sequence of wages are identical to the ones obtained from the social planner's problem then, we can state that the retirement age resulting from the agent's problem is precisely the optimal one determined by the social planner[18].

The return of assets is contingent in the age of the agent, $r(a,t)$, and satisfies the arbitrage condition (8.10):

$$r(a,t)=\begin{cases} r(t) & \text{if } a\le a_o \\ r(t)-\frac{\dot{s}(a)}{s(a)} & \text{otherwise}\end{cases} \tag{8.16}$$

The Euler equation (8.13) together with equation (8.16) imply the following Euler equation:

$$\frac{\dot{c}(t)}{c(t)}=[r(t)-\rho] \Longleftrightarrow \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)}=[r(t)-\rho-\gamma] \tag{8.17}$$

where $\widetilde{c}(t)$ is defined in Section 4 as $\widetilde{c}(t)\equiv\frac{c(t)}{\Gamma(t)}$. Notice that the Euler equation defined by equation (8.17) is exactly the same as in the optimal allocation (see equation 8.6).

[18]Even in the case that taxes were distortionary instead of lump sump (for instance, a proportional tax to the labor income), it is always possible to induce the agent to retire at the optimal age by introducing a penalty in the future pensions when she retires before reaching the optimal retirement age. We illustrate this idea in example 8.2.

The budget constraint of an agent born at $t_b$ (equation 8.11) together with condition (8.14) imply the following intertemporal budget constraint:

$$\int_{t_b+a_y}^{t_b+2a_y} \left[1+e^{na_y}\right]\widetilde{c}(\tau)e^{\int_{t_b+a_y}^{\tau}[r(x)-\gamma]dx}d\tau + \int_{t_b+2a_y}^{t_b+\overline{a}} \widetilde{c}(\tau)e^{\int_{t_b+a_y}^{\tau}[r(x)-\gamma]dx}d\tau =$$
$$\int_{t_b+a_y}^{t_b+a_r}\left[\widetilde{w}(\tau)h(\tau\text{-}t_b)\text{-}\widetilde{T}(\tau\text{-}t_b,\tau)\right]e^{\int_{t_b+a_y}^{\tau}[r(x)\text{-}\gamma]dx}d\tau + \int_{t_b+a_r}^{t_b+\overline{a}} \widetilde{tr}(\tau\text{-}t_b,\tau)e^{\int_{t_b+a_y}^{\tau}[r(x)\text{-}\gamma]dx}d\tau$$

The intertemporal budget constraint shows that the present value of the household's consumption, including the consumption of children when they are infants, should be equal to the present value of the life-time disposable income of the agent, that is, wages minus taxes plus transfers. It follows from the optimal condition of the retirement age (equation 8.12) and the Euler Equation (equation 8.17) that in order that agents consume the same amount as in the optimal consumption path when prices are the optimal ones, the following two necessary and sufficient conditions should hold $\forall t \in \mathbb{R}$:

**Condition (1):**

$$\int_{t_b+a_y}^{t_b+2a_y} \left[1+e^{na_y}\right]\widetilde{c}^{*}(\tau)e^{\int_{t_b+a_y}^{\tau}[r^{*}(x)-\gamma]dx}d\tau + \int_{t_b+2a_y}^{t_b+\overline{a}} \widetilde{c}^{*}(\tau)e^{\int_{t_b+a_y}^{\tau}[r^{*}(x)-\gamma]dx}d\tau =$$
$$\int_{t_b+a_y}^{t_b+a_r^{*}}\left[\widetilde{w}^{*}(\tau)h(\tau\text{-}t_b)\text{-}\widetilde{T}(\tau\text{-}t_b,\tau)\right]e^{\int_{t_b+a_y}^{\tau}[r^{*}(x)\text{-}\gamma]dx}d\tau + \int_{t_b+a_r^{*}}^{t_b+\overline{a}} \widetilde{tr}(\tau\text{-}t_b,\tau)e^{\int_{t_b+a_y}^{\tau}[r^{*}(x)\text{-}\gamma]dx}d\tau$$

**Condition (2):**

$$-\int_{a_y}^{a_r^{*}} \mu(a)\widetilde{T}(a,t)da + \int_{a_r^{*}}^{\overline{a}} \mu(a)\widetilde{tr}(a,t)da = 0.$$

If Condition (1) is satisfied then, the optimal consumption path chosen by the agent, $\widetilde{c}^{*}$, satisfies the budget constraint with equality. Due to the fact that first order conditions are exactly the same as in the optimal solution (see equations 8.17 and 8.12), it is easy to prove that the consumption path chosen by the agent coincides with the optimal consumption path chosen by the social planner. This implies that the capital path also coincides with the optimal capital path and consequently, equilibrium prices coincide with optimal ones. Condition (2) simply means that the budget constraint of the government is satisfied with equality every period, this is, the social security system is defined as a pay as you go system. Therefore, any social security system that satisfies these two conditions is an optimal one, since the resulting equilibrium allocations would be the same as the ones that can be obtained from the social planner problem.

### 8.2.1. Optimal Social Security at the steady state

The optimal social security system at the steady state should satisfy the following two conditions:

**Condition (1):**

$$\int_{a_y}^{2a_y} \left[1+e^{na_y}\right] \widetilde{c}^{*} e^{\rho(\tau-a_y)} d\tau + \int_{2a_y}^{\overline{a}} \widetilde{c}^{*} e^{\rho(\tau-a_y)} d\tau =$$

$$\int_{a_y}^{a_r^*} \left[\widetilde{w}^{*} h(a) - \widetilde{T}(a)\right] e^{\rho(\tau-a_y)} d\tau + \int_{a_r^*}^{\overline{a}} \widetilde{tr}(a) e^{\rho(\tau-a_y)} d\tau$$

**Condition (2):**

$$-\int_{a_y}^{a_r^*} \mu(a)\widetilde{T}(a)da + \int_{a_r^*}^{\overline{a}} \mu(a)\widetilde{tr}(a)da \; = \; 0$$

Since satisfying conditions (1) and (2) are the unique requirement that a social security system should satisfy for being an optimal one, it is possible to find many alternative optimal systems. Next, we show two examples of social security systems that are optimal ones.

**Example 8.1.** ***Lump-sum taxes:*** *Let's assume that taxes, $T$, grow at rate $\gamma$ along the working life, $\forall a \in [a_y, a_r]$ $\widetilde{T}(a) = \widetilde{T}$, this is, $\widetilde{T}$ is constant along working life, and pensions grow at rate $\gamma$ during retirement, $\forall a \in [a_r, \overline{a}]$ $\widetilde{tr}(a) = \widetilde{tr}$ ($\widetilde{tr}$ is* *constant). In this case, following conditions should hold:*

$$\widetilde{c}^* \left[\left[1+e^{na_y}\right]\left[e^{\rho a_y}-1\right]+\left[e^{\rho(\overline{a}-a_y)}-e^{\rho a_y}\right]\right] =$$
$$\widetilde{w}^* \int_{a_y}^{a_r^*} h(a)e^{\rho(\tau-a_y)}d\tau - \widetilde{T}\left[e^{\rho(a_r^*-a_y)}-1\right]+\widetilde{tr}\left[e^{\rho(\overline{a}-a_y)}-e^{\rho(a_r^*-a_y)}\right]$$
$$-\widetilde{T}\int_{a_y}^{a_r^*}\mu(a)da+\widetilde{tr}\int_{a_r^*}^{\overline{a}}\mu(a)da = 0$$

*These equations imply the following tax-transfer scheme:*

$$\widetilde{tr}^* = \frac{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[\widetilde{w}^*\int_{a_y}^{a_r*}h(a)e^{\rho(\tau-a_y)}d\tau-\widetilde{c}^*\left[e^{na_y}\left[e^{\rho a_y}-1\right]+e^{\rho(\overline{a}-a_y)}-1\right]\right]}{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[e^{\rho(\overline{a}-a_y)}-e^{\rho a_y}\right]-\left[\int_{a_r^*}^{\overline{a}}\mu(a)da\right]\left[e^{(a_r^*-a_y)}-1\right]}$$

$$\widetilde{T}^* = \frac{\left[\int_{a_r^*}^{\overline{a}}\mu(a)da\right]\left[\widetilde{w}^*\int_{a_y}^{a_r^*}h(a)e^{\rho(\tau-a_y)}d\tau-\widetilde{c}^*\left[e^{na_y}\left[e^{\rho a_y}-1\right]+e^{\rho(\overline{a}-a_y)}-1\right]\right]}{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[e^{\rho(\overline{a}-a_y)}-e^{\rho a_y}\right]-\left[\int_{a_r^*}^{\overline{a}}\mu(a)da\right]\left[e^{\rho(a_r^*-a_y)}-1\right]}$$

**Example 8.2.** ***Distortionary taxes:*** *Let's consider now a tax proportional to the labor income. Since this kind of taxes increases the incentive to retire early, we introduce a penalty for early retirement in order to overcome it. Thus, we guarantee that we are considering a neutral or an actuarially fair system. To do that, we consider that there exists a proportional tax to labor income with tax rate $\tau$ and a retirement pension with a penalty in case of early retirement. More precisely, the retirement pension would be a continuous differentiable increasing function of the retirement age, $tr(a_r)$, which means that the pension increases with the retirement age. The first order condition of the household's problem with respect to the retirement age with this tax-pension scheme would be as follows:* [19]

$$\phi(a_r) = \frac{1}{\widetilde{c}(t_b+a_r)}\left[\widetilde{w}(t_b+a_r)h(a_r)(1-\tau)-\widetilde{tr}(a_r,t_b+a_r)\right]+$$
$$\int_{t_b+a_r}^{t_b+\overline{a}}\frac{1}{\widetilde{c}(t)}e^{-\rho(t-(t_b+a_r))}\left[\widetilde{tr}'(a_r,t)\right]dt \tag{8.18}$$

[19]See appendix for details.

*At the steady state with an optimal social security system (a social security that replicate the social planner allocation), this condition would be as follows:*

$$\begin{aligned} \phi(a_r^*) &= \frac{1}{\widetilde{c}^*}\widetilde{w}^*\left(1-\tau\right)h(a_r^*) - \widetilde{tr}(a_r^*) + \frac{1}{\widetilde{c}^*}\widetilde{tr}'(a_r^*)\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right] \Leftrightarrow \\ \frac{\phi(a_r^*)\widetilde{c}^*}{h(a_r^*)} - \widetilde{w}^* &= -\tau\widetilde{w}^* - \widetilde{tr}(a_r^*) + \frac{\widetilde{tr}'(a_r^*)}{h(a_r^*)}\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right] \end{aligned} \tag{8.19}$$

*If* $\frac{\widetilde{tr}'(a_r^*)}{h(a_r^*)}$ *is strictly decreasing* [20] *and* $\tau\widetilde{w}^* + \widetilde{tr}(a_r^*) = \frac{\widetilde{tr}'(a_r^*)}{h(a_r^*)}\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right]$ *then the agent will chose the optimal retirement age if the sequence of prices are the one in the optimal allocation. Thus, if the penalty for early retirement is large enough, it is possible to induce the household to choose the optimal retirement age (see for example the transfer function* **??***). Notice that if agents would own the wealth level which implies the same path of consumption as in the optimal allocation, then prices will be the same as the ones in the optimal allocation. For having this, the tax rate and retirement pensions should satisfy the following two conditions:*

$$\begin{aligned} &\widetilde{c}^*\left[\left[1+e^{na_y}\right]\left[e^{\rho a_y}-1\right]+\left[e^{\rho(\overline{a}^*-a_y)}-e^{\rho a_y}\right]\right] = \\ &\widetilde{w}^*(1-\tau)\int_{a_y}^{a_r^*} h(a)e^{\rho(\tau-a_y)}d\tau + \widetilde{tr}(a_r^*)\left[e^{\rho(\overline{a}^*-a_y)}-e^{\rho(a_r^*-a_y)}\right] \\ &-\tau\widetilde{w}^*\int_{a_y}^{a_r^*}\mu(a)h(a)da + \widetilde{tr}(a_r^*)\int_{a_r^*}^{\overline{a}}\mu(a)da = 0 \end{aligned}$$

*which imply the following tax-pension scheme:*

$$\widetilde{tr}^*(a_r^*) = \frac{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[\widetilde{w}^*\int_{a_y}^{a_r^*}h(a)e^{\rho(\tau-a_y)}d\tau - \widetilde{c}^*\left[e^{na_y}\left[e^{\rho a_y}-1\right]+e^{\rho(\overline{a}-a_y)}-1\right]\right]}{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[e^{\rho(\overline{a}-a_y)}-e^{\rho a_y}\right]-\left[\int_{a_r^*}^{\overline{a}}\mu(a)da\right]\left[e^{\rho(a_r^*-a_y)}-1\right]} \tag{8.20}$$

$$\tau^* = \frac{\left[\int_{a_r^*}^{(\overline{a}-a_y)}\mu(a)da\right]\left[\int_{a_y}^{a_r^*}h(a)e^{\rho(\tau-a_y)}d\tau - \frac{\widetilde{c}^*}{\widetilde{w}^*}\left[e^{na_y}\left[e^{\rho a_y}-1\right]+e^{\rho(\overline{a}-a_y)}-1\right]\right]}{\left[\int_{a_y}^{a_r^*}\mu(a)da\right]\left[e^{\rho(\overline{a}-a_y)}-e^{\rho a_y}\right]-\left[\int_{a_r^*}^{\overline{a}}\mu(a)da\right]\left[e^{\rho(a_r^*-a_y)}-1\right]} \tag{8.21}$$

[20] Assuming that function $\frac{\widetilde{tr}'(a_r^*)}{h(a_r^*)}$ is strictly decreasing is a sufficient condition to guarantee that the Hamiltonian is concave with respect to $a_r$. Furthermore, it guarantees that equation (8.19) has a unique solution (for a give consumption).

*An example of transfer function would be as follows:*

$$\widetilde{tr}(a_r) \;=\; \left[\widetilde{tr}^{\,*}(a_r^*)\right] + \left(a_r^{\theta} - (a_r^*)^{\theta}\right)\xi$$

*where* $\xi$ *is a positive constant defined according to equation (8.19), this is,*

$$\begin{aligned}
\xi \overset{Def}{\Leftrightarrow} \tau\widetilde{w}^* + \widetilde{tr}(a_r^*) \;&=\; \frac{\widetilde{tr}'(a_r^*)}{h(a_r^*)}\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right] \;\Leftrightarrow\\
\tau\widetilde{w}^* + \widetilde{tr}(a_r^*) \;&=\; \frac{\xi}{h(a_r^*)}\frac{\theta}{(a_r^*)^{\theta}}\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right] \;\Leftrightarrow\\
\xi \;&\equiv\; \frac{\left[\tau\widetilde{w}^* + \widetilde{tr}(a_r^*)\right](a_r^*)^{\theta}}{\theta\left[\frac{e^{-\rho(\overline{a}-a_r^*)}-1}{\rho}\right]h(a_r^*)}
\end{aligned}$$

*and* $\widetilde{tr}^{\,*}(a_r^*)$ *is as defined in (8.20) and* $\theta \in (0,1)$.

## 9. Conclusion

The continuous fall in fertility and the rise in longevity experienced during recent decades in developed countries have led to a significant increase in the proportion of the older population and a decrease in the number of working-age people for every elderly person. This increase in the dependency ratio has aroused the debate of economists and politicians about the introduction of policy measures, such as fertility enhancing programs or delaying the retirement age.

In this paper, we formulate a growth model which determines the optimal retirement age that economies should implement in order to reach the optimal size of the work force relative to the whole population. Then, we use the model to analyze the effect of an ageing population on the optimal retirement age. We have considered two possible sources of aging: an increase in the life expectancy and a decrease in the fertility rate, and we made clear that these sources of ageing population have different effects. In fact, under empirically plausible assumptions, these two sources of aging have exactly opposite effects. More precisely, an increase in life expectancy extends the optimal retirement age at the steady state; while a drop in the fertility rate diminishes the optimal retirement age at the steady state.

Keeping the retirement age fixed, a lengthening of life expectancy raises the portion of old agents in the population, implying an increase of the dependency ratio. This fact produces a negative "wealth effect" in the economy which in turn reduces per capita consumption and enlarges the retirement age at the steady state. The effect of a fall in the fertility rate is more complex. A drop in the fertility rate decreases the portion of children and increases the share of the retired population. Thus, the effect on the activity rate and the dependency ratio is ambiguous at the first glance. However, we show that if the average age of workers is larger than the average age of total population, a fall in the fertility rate raises the activity rate (and diminishes the dependency ratio). However, an additional mechanism should be considered in order to obtain the whole effect of a drop in fertility over per capita labor. Due to the fact that the productivity of workers varies along the working life with age, the change in the weights of workers of different ages affects the aggregate productivity of labor. If for instance, older workers are more experienced and productive than the average, a drop in the fertility rate, which gives more weight to these workers, would imply an increase in labor productivity and per capita labor. Nevertheless, even taking into account this additional mechanism, we offer a clear yardstick to determine the net effect of a fall in fertility over per capita labor: if the average age of labor units is higher than the average age of total population, then a drop in fertility would imply an increase in per capita labor, and the optimal response for the economy is a reduction in the retirement age.

We calculate these key statistics involved in our results using empirical data from a wide sample of developed countries. Using our theoretical outcomes and these statistics, we determine the consequences of a drop in the fertility rate over the activity rate, the per capita efficiency units of labor and the optimal retirement age in those countries. Our results are running against the common wisdom: we find that a drop in fertility would increase the per capita efficiency units of labor and would reduce the optimal retirement age in the vast majority of the countries in the sample. Thus, our findings suggest that developed countries that are involved in pronatalist policies targeted to avoid the delay of the retirement age and to overcome the negative effect of ageing population over the effective labor supply, are not choosing the right solution.

Moreover, we incorporate exogenous growth in the model in order to study the effect of economic growth on the optimal retirement age. In spite of the offsetting mechanisms that are involved, we have established a clear criterion to determine the effect of growth rate in the retirement age. More precisely, under plausible conditions from an empirical point of view, a fall in the long run growth rate (in the rate of technological change) involves an increase in the optimal retirement age. Thus, the fall in the growth rate that developed countries have suffered in the last decades is another factor to take into account in the retirement age debate.

Finally, we show how to decentralize the social planner efficient allocation through a social security system.

This paper pushes forward the importance of two elements which are key to clarify the ageing population debate and which are often overlooked: first, the need to consider the whole demographic structure of the population in order to analyze the effect of demographic political measures on the labor force. More precisely, the paper reveals the relevance of studying not only the effect on elder population but also the effect on other inactive groups in the labor market, like the non-adult population. The second element to be considered is the productivity profile along the working life. This profile may affect decisively on the effect of demographic policies on the labor force, since such policies may alter the composition of different demographic groups (with different productivity levels) inside the labor force, and consequently, may alter the labor productivity. The consideration of these two elements is essential to correctly design policy measures.

# 10. Appendix

In what follows we omit the time label, $(t)$, whenever this does not generate confusion.

## 10.1. The per capita labor supply

In this subsection, we analyze the effect of different exogenous variables on the labor supply. The per capita labor supply is as follows:

$$l(a_r) = \int_{a_y}^{a_r} h(a)\mu(a)da = \frac{\int_{a_y}^{a_r} h(a)s(a)e^{-na}da}{\left[\int_0^{\overline{a}} s(a)e^{-na}da\right]} = \frac{\int_{a_y}^{a_r} h(a)e^{-na}da}{\left[\int_0^{a_o} e^{-na}da\right] + \left[\int_{a_o}^{\overline{a}} s(a,\xi)e^{-na}da\right]}$$

$$\frac{\partial l(a_r)}{\partial a_r} = h(a_r)\mu(a_r) > 0$$

The effect of increasing life expectancy (increase $\xi$) is to reduce per capita labor supply:

$$\frac{\partial l(a_r)}{\partial \xi} = -l(a_r)\frac{\left[\int_{a_o}^{\overline{a}} s'_{\xi}(a,\xi)e^{-na}da\right]}{\left[\left[\int_0^{a_o} e^{-na}da\right] + \left[\int_{a_o}^{\overline{a}} s(a,\xi)e^{-na}da\right]\right]^2} < 0 \tag{10.1}$$

The effect of an increase in the fertility rate on the labor supply is not clear for two reasons. First, it reduces the portion of young individuals while increases the portion of old individuals, implying an ambiguous result on the work force, and; second, if labor productivity increases with age ($\frac{\partial h(a)}{\partial a} > 0$) then, the increase of the fertility rate reduces the portion of highly productive agents while increases the portion of the less productive ones, therefore producing, again, an ambiguous result on the labor supply:

$$\begin{aligned}
\frac{\partial l(a_r)}{\partial n} &= l(a_r)\left[\frac{\int_0^{\overline{a}} a\ s(a)e^{-na}da}{\int_0^{\overline{a}} s(a)e^{-na}da} - \frac{\int_{a_y}^{a_r} a\ h(a)s(a)e^{-na}da}{\int_{a_y}^{a_r} s(a)h(a)e^{-na}da}\right] = \\
&= l(a_r)\left[\left(\int_0^{\overline{a}} a\ \frac{s(a)e^{-na}}{\left[\int_{a_y}^{a_r} s(a)e^{-na}da\right]}da\right) - \left(\int_{a_y}^{a_r} a\ \frac{h(a)s(a)e^{-na}}{\left[\int_{a_y}^{a_r} h(a)s(a)e^{-na}da\right]}da\right)\right] \\
&= l(a_r)\left(E\left[a\right] - E_{\nu}\left[a/\left[a_y, a_r\right]\right]\right)
\end{aligned} \tag{10.2}$$

Thus, the sign of the above derivative depends on the relationship between the average age of the population, $E[a]$, and the average age of labor units, $E_{\nu}[a/[a_y, a_r]]$.

### 10.2. The steady state

This section describes the construction of the steady state. We first calculate and study the two loci ($\dot{a}_r = 0$ and $\dot{\widetilde{k}}_r = 0$) and then, we solve the resulting equations system. We prove that the solution is unique and so, there is only one steady state.

#### 10.2.1. Locus $\dot{a}_r = 0$

The locus in which the retirement age remains constant is as follows:

$$\left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} = \left[\frac{(1-\alpha)\alpha}{[\rho + (1-\alpha)(\delta+\gamma) - \alpha n]} \frac{h(a_r)}{\phi(a_r)[l(a_r)]^{\alpha}}\right]^{\frac{1}{1-\alpha}}$$

The slope of the above locus is:

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial a_r} = -\frac{1}{1-\alpha} \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} \left[\frac{\frac{\partial\left(\frac{\phi(a_r)}{h(a_r)}\right)}{\partial a_r}}{\left(\frac{\phi(a_r)}{h(a_r)}\right)} + \alpha\frac{\mu(a)h(a_r)}{l(a_r)}\right] < 0$$

given that we have assumed that $\frac{\phi(a)}{h(a)}$ is an increasing function in $a$.

Furthermore, we can obtain the effect of different exogenous variables:

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial \xi} = -\frac{\alpha}{1-\alpha} \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} \frac{\frac{\partial l(a_r)}{\partial \xi}}{l(a_r)} > 0$$

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial n} = \frac{\alpha}{1-\alpha} \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} \left[-\frac{\frac{\partial l(a_r)}{\partial n}}{l(a_r)} + \frac{1}{[\rho + (1-\alpha)(\delta+\gamma) - \alpha n]}\right] =$$

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial n} = \frac{\alpha}{1-\alpha} \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} \left[-\left(E[a] - E_{\nu}[a/[a_y, a_r]]\right) + \frac{1}{[\rho + (1-\alpha)(\delta+\gamma) - \alpha n]}\right]$$

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial \gamma} = -\frac{\alpha}{1-\alpha} \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0} \left[\frac{1}{[\rho + (1-\alpha)(\delta+\gamma) - \alpha n]}\right] < 0$$

where we have used equation (10.2) in the derivative $\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{a}_r=0}}{\partial n}$.

### 10.2.2. Locus $\dot{\widetilde{k}} = 0$:

$$\left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0} = l(a_r)\left[\frac{\left[1 - \frac{(1-\alpha)}{\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]}{(\delta+n+\gamma)}\right]^{\frac{1}{1-\alpha}}$$

The slope of the above locus is:

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}}{\partial a_r} = \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}\left[\frac{\frac{\partial l(a_r)}{\partial a_r}}{l(a_r)} + \frac{\frac{\frac{\partial\left(\frac{\phi(a_r)}{h(a_r)}\right)}{\partial a_r}}{\frac{\phi(a_r)}{h(a_r)}} + \frac{\frac{\partial l(a_r)}{\partial a_r}}{l(a_r)}}{\left[1 - \frac{(1-\alpha)}{\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right] > 0$$

Furthermore, we can obtain the effect of different exogenous variables:

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}}{\partial \xi} = \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}\left[1 + \frac{1}{\left[1 - \frac{(1-\alpha)}{\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\frac{\partial l(a_r)}{\partial \xi}}{l(a_r)} < 0$$

$$\begin{aligned}\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}}{\partial n} &= \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}\left[\left[1 + \frac{1}{\left[1 - \frac{(1-\alpha)}{\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\frac{\partial l(a_r)}{\partial n}}{l(a_r)} - \frac{1}{1-\alpha}\frac{1}{(\delta+n+\gamma)}\right] \\ &= \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}\left[\left[1 + \frac{1}{\left[1 - \frac{(1-\alpha)}{\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right]\frac{\phi(a_r)}{h(a_r)}l(a_r)}\right](E[a] - E_\nu[a/[a_y, a_r]]) - \frac{1}{1-\alpha}\frac{1}{(\delta+n+\gamma)}\right]\end{aligned}$$

$$\frac{\partial \left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}}{\partial \gamma} = -\left.\widetilde{k}(a_r)\right|_{\dot{\widetilde{k}}=0}\frac{1}{1-\alpha}\frac{1}{(\delta+n+\gamma)} < 0$$

### 10.2.3. Steady State

It follows from the two loci analyzed in the above section that the retirement age at the steady state should satisfy the following equation:

$$\begin{aligned}
\left[\frac{(1-\alpha)\alpha}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\frac{h(a_r^{ss})}{\phi(a_r^{ss})\left[l(a_r^{ss})\right]^{\alpha}}\right]^{\frac{1}{1-\alpha}} &= l(a_r^{ss})\left[\frac{\left[1-(1-\alpha)\frac{h(a_r^{ss})}{\phi(a_r^{ss})l(a_r^{ss})}\right]}{(\delta+n+\gamma)}\right]^{\frac{1}{1-\alpha}} \\
\frac{(1-\alpha)\alpha(\delta+n+\gamma)}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\frac{h(a_r^{ss})}{\phi(a_r^{ss})l(a_r^{ss})} &= \left[1-(1-\alpha)\frac{h(a_r^{ss})}{\phi(a_r^{ss})l(a_r^{ss})}\right] \\
\frac{\phi(a_r^{ss})l(a_r^{ss})}{h(a_r^{ss})} &= \frac{(1-\alpha)\left[\alpha(\delta+n+\gamma)+1\right]}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}
\end{aligned} \tag{10.3}$$

It follows from Euler equation (3.5) that at the steady state:

$$\alpha\left(\frac{\widetilde{k}^{ss}}{l(a_r^{ss})}\right)^{\alpha-1} = \delta+\rho+\gamma \Rightarrow \widetilde{k}^{ss} = \left(\frac{\alpha}{\delta+\rho+\gamma}\right)^{\frac{1}{1-\alpha}} l(a_r^{ss}) \tag{10.4}$$

Using equations (10.3) and (10.4) we get an alternative expression for per capita capital:

$$\widetilde{k}^{ss} = \left(\frac{\alpha}{\delta+\rho+\gamma}\right)^{\frac{1}{1-\alpha}}\frac{(1-\alpha)\left[\alpha(\delta+n+\gamma)+1\right]}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\left[\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right]^{-1} \tag{10.5}$$

Using the above equation and (3.4):

$$\frac{\phi(a_r^{ss})}{h(a_r^{ss})} = \frac{1}{\widetilde{c}^{ss}}(1-\alpha)\left(\frac{\widetilde{k}^{ss}}{l(a_r^{ss})}\right)^{\alpha} \Rightarrow \widetilde{c}^{ss} = \frac{(1-\alpha)\left(\frac{\alpha}{\delta+\rho+\gamma}\right)^{\frac{\alpha}{1-\alpha}}}{\frac{\phi(a_r^{ss})}{h(a_r^{ss})}} \tag{10.6}$$

### 10.3. Proof of proposition 5.1

It follows from equations (10.3), (10.1) and the Implicit Function Theorem that:

$$\frac{\partial a_r^{ss}}{\partial \xi} = -\frac{\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial \xi}}{\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}}l(a_r^{ss})+\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial a_r^{ss}}} > 0$$

It follows from equations (10.6) and (10.5) that:

$$\begin{aligned}
\frac{\partial \widetilde{c}^{ss}}{\partial \xi} &= -\widetilde{c}^{ss}\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}}\frac{\partial a_r^{ss}}{\partial \xi} < 0 \\
\frac{\partial \widetilde{k}^{ss}}{\partial \xi} &= -\widetilde{k}^{ss}\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}}\frac{\partial a_r^{ss}}{\partial \xi} < 0
\end{aligned}$$

### 10.4. Proof of proposition 5.2:

Using the definition of the density function $\mu(.)$ (equation 2.6) and the definition of activity rate, equation (5.1):

$$AR = \int_{a_y}^{a_r} \mu(a)da = \frac{\int_{a_y}^{a_r} s(a)e^{-na}da}{\int_0^{\overline{a}} s(a)e^{-na}da}$$

$$\frac{\partial AR}{\partial n} = AR\left[\frac{\int_0^{\overline{a}} a\, s(a)e^{-na}da}{\int_0^{\overline{a}} s(a)e^{-na}da} - \frac{\int_{a_y}^{a_r} a\, s(a)e^{-na}da}{\int_{a_y}^{a_r} s(a)e^{-na}da}\right] = AR\left[E\left[a\right] - E\left[a/\left[a_y, a_r\right]\right]\right]$$

■

### 10.5. Proof of proposition 5.3

See equation (10.2). ■

### 10.6. Proof of proposition 5.5

It follows from equations (10.3), (10.2) and the Implicit Function Theorem that:

$$\begin{aligned}
\frac{\partial a_r^{ss}}{\partial n} &= -\frac{\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial n} - \frac{(1-\alpha)[\alpha(\delta+n+\gamma)+1]}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\left[\frac{\alpha}{\alpha(\delta+n+\gamma)+1} + \frac{\alpha}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\right]}{\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}} l(a_r^{ss}) + \frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial a_r^{ss}}} = \\
&= \frac{\frac{\phi(a_r^{ss}) l(a_r)}{h(a_r^{ss})}\left[-\left(E\left[a\right] - E_\nu\left[a/\left[a_y, a_r\right]\right]\right) + \frac{\alpha(1+\rho+\delta+\gamma)}{[\alpha(\delta+n+\gamma)+1][\rho+(1-\alpha)(\delta+\gamma)-\alpha n]}\right]}{\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}} l(a_r^{ss}) + \frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial a_r^{ss}}} \\
&= \frac{\frac{\alpha[1+\delta+\rho+\gamma]}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n][1+\alpha(\delta+\rho+\gamma)]} + \left(E_\nu\left[a/\left[a_y, a_r(t)\right]\right] - E\left[a\right]\right)}{\frac{\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}}}{\frac{\phi(a_r^{ss})}{h(a_r^{ss})}} + \frac{\mu(a_r^{ss})h(a_r^{ss})}{l(a_r^{ss})}} > 0
\end{aligned}$$

It follows from (10.6) and (10.5) that:

$$\frac{\partial \widetilde{c}^{ss}}{\partial n} = -\widetilde{c}^{ss}\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}}\frac{\partial a_r^{ss}}{\partial n} < 0$$

■

### 10.7. Proof of proposition 6.1

It follows from (10.3) and the Implicit Function Theorem that:

$$\frac{\partial a_r^{ss}}{\partial \gamma} = \frac{\frac{(1-\alpha)[\alpha(\rho-n)-(1-\alpha)]}{[\rho+(1-\alpha)(\delta+\gamma)-\alpha n]^2}}{\frac{\partial\left(\frac{\phi(a_r^{ss})}{h(a_r^{ss})}\right)}{\partial a_r^{ss}} l(a_r^{ss}) + \frac{\phi(a_r^{ss})}{h(a_r^{ss})}\frac{\partial l(a_r^{ss})}{\partial a_r^{ss}}}$$

### 10.8. Example 8.2: Household's maximization problem with an optimal social security system with distortionary taxes

$$\begin{aligned}
&\max_{[c(t)]_{t_b+a_y}^{t_b+\overline{a}},[c_{ch}(t)]_{t_b+a_y}^{t_b+2a_y},a_r} \int_{t_b+a_y}^{t_b+2a_y} \left[\ln(c(t)) + e^{na_y}\ln(c_{ch}(t))\right] e^{-\rho(t-(t_b+a_y))}dt + \\
&\int_{t_b+2a_y}^{t_b+a_o} \ln(c(t))e^{-\rho(t-(t_b+a_y))}dt + \int_{t_b+a_o}^{t_b+\overline{a}} \ln(c(t))s(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt - \\
&\int_{t_b+a_y}^{t_b+a_r} \phi(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt + \lambda(t) \\
&\text{s.t.:} \qquad (10.7) \\
&\dot{b}(t) = \\
&\begin{cases} w(t)h(t\text{-}t_b)(1-\tau)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t)\text{-}e^{na_y}c_{ch}(t) & \text{if } t\in[t_b+a_y,t_b+2a_y] \\ w(t)h(t\text{-}t_b)(1-\tau)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t) & \text{if } t\in[t_b+2a_y,t_b+a_r] \\ tr(a_r,t)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t) & \text{if } t\in[t_b+a_r,\overline{a}] \end{cases} \\
&b(t_b+a_y) = 0;\ b(t_b+\overline{a}) \geq 0
\end{aligned}$$

The Hamilitonian function associated to the above problem would be as follows:

$$\begin{aligned}
&\int_{t_b+a_y}^{t_b+2a_y} \left[\ln(c(t)) + e^{na_y}\ln(c_{ch}(t))\right] e^{-\rho(t-(t_b+a_y))}dt + \\
&\int_{t_b+2a_y}^{t_b+a_o} \ln(c(t))e^{-\rho(t-(t_b+a_y))}dt + \int_{t_b+a_o}^{t_b+\overline{a}} \ln(c(t))s(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt - \\
&\int_{t_b+a_y}^{t_b+a_r} \phi(t\text{-}t_b)e^{-\rho(t-(t_b+a_y))}dt + \\
&\int_{t_b+a_y}^{t_b+2a_y} \lambda(t)\left[w(t)h(t\text{-}t_b)(1-\tau)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t)\text{-}e^{na_y}c_{ch}(t)\right]dt + \qquad (10.8) \\
&\int_{t_b+2a_y}^{t_b+a_r} \lambda(t)\left[w(t)h(t\text{-}t_b)(1-\tau)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t)\right]dt + \qquad (10.9) \\
&\int_{t_b+a_r}^{t_b+\overline{a}} \lambda(t)\left[tr(a_r,t)+r\,(t\text{-}t_b,t)\,b(t)\text{-}c(t)\right]dt \qquad (10.10)
\end{aligned}$$

First order conditions are:

$$\frac{1}{c(t)}e^{-\rho(t-(t_b+a_y))} = \lambda(t) \tag{10.11}$$

$$\frac{1}{c_{ch}(t)}e^{na_y}e^{-\rho(t-(t_b+a_y))} = \lambda(t)e^{na_y} \tag{10.12}$$

$$\phi(a_r)e^{-\rho(a_r-a_y)} = \lambda(t_b{+}a_r)\left[w(t_b{+}a_r)h(a_r)(1-\tau) - tr(a_r, t_b{+}a_r)\right] + \int_{t_b+a_r}^{t_b+\overline{a}} \lambda(t)\left[tr'(a_r,t)\right]dt \tag{10.13}$$

$$\dot{\lambda}(t) = \lambda(t)r\,(t\text{-}t_b, t) \tag{10.14}$$

Conditions (10.11) and (10.13) yields:

$$\phi(a_r) = \frac{1}{c(t_b{+}a_r)}\left[w(t_b{+}a_r)h(a_r)(1-\tau) - tr(a_r, t_b{+}a_r)\right] + \int_{t_b+a_r}^{t_b+\overline{a}} \frac{1}{c(t)}e^{-\rho(t-(t_b+a_r))}\left[tr'(a_r,t)\right]dt \tag{10.15}$$

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